MATH2050C Selected Solution to Assignment 9

Section 4.3

(3) Let $M > 0$ be given. We take $\delta = 1/M^2$. Then for $x, 0 < |x| < \delta$,

$$\frac{1}{\sqrt{x}} > \frac{1}{\sqrt{\delta}} = M \ ,$$

so $\lim_{x\to 0} 1/\sqrt{|x|} = \infty$.

(5a) The right hand limit is ∞. (More precisely, the right hand limit does not exist; it diverges to ∞.) Let $M > 0$. Choose $\delta = 1/M$. Then for $x, 1 < x < 1 + 1/M$,

$$\frac{x}{x-1} \geq \frac{1}{x-1} > M \ .$$

(5b) The limit does not exist. The right limit diverges to ∞ and the left limit diverges to $-\infty$. For $M > 0$. Choose $\delta = \min\{1/2, 1/(2M)\}$. Then for $x, 0 < |x - 1| < \delta$, we have $x > 1 - \delta \geq 1 - 1/2 = 1/2$. Thus,

$$\frac{x}{x-1} \geq \frac{1/2}{x-1} > \frac{1}{2\delta} \geq \frac{1}{2 \times M/2} = M \ .$$

(5h) The limit exists and is equal to -1. For $x > 0$,

$$0 \leq \left| \frac{\sqrt{x} - x}{\sqrt{x} + x} - (-1) \right| = \left| \frac{2\sqrt{x}}{\sqrt{x} + x} \right| \leq \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}} \ .$$

By Squeeze Theorem,

$$0 \leq \lim_{x\to\infty} \frac{\sqrt{x} - x}{\sqrt{x} + x} \leq \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0 \ .$$

Supplementary Exercises

Justify your answers in the following problems.

1. Evaluate

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2}.$$

Solution. Use $(x^2 - 4)/(x - 2) = (x - 2)(x + 2)/(x - 2) = x + 2$ for $x \neq 2$, we have

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2}(x + 2) = 4.$$

2. Evaluate
$$\lim_{x\to -3}\frac{x^2-2x-15}{x+3}.$$

Solution. Use $x^2-2x-15=(x+3)(x-5)$, we have
$$\lim_{x\to -3}\frac{x^2-2x-15}{x+3}=\lim_{x\to -3}(x-5)=-8.$$

3. Evaluate
$$\lim_{x\to\infty}\frac{\cos x}{x}.$$

Observe that $|\cos x|\le 1$ for all x. For $\varepsilon>0$, take $K=1/\varepsilon$. Then
$$\left|\frac{\cos x}{x}\right|\le\frac{1}{x}<\varepsilon\ ,\quad \forall x>K.$$
We conclude that the limit is equal to 0.

4. Evaluate
$$\lim_{x\to x_0}\frac{\sin x-\sin x_0}{x-x_0}\ .$$

Hint: Let $h=x-x_0$ and reduce the problem to $h\to 0$. Then make use of the compound angle formula for the sine function.

Solution.
$$\lim_{x\to x_0}\frac{\sin x-\sin x_0}{x-x_0}=\lim_{h\to 0}\frac{\sin(x_0+h)-\sin x_0}{h}\ .$$
Use the compound angle formula, $\sin(x_0+h)=\sin x_0\cos h+\cos x_0\sin h$ and the known limits $\lim_{h\to 0}\sin h/h=1$ and $\lim_{h\to 0}(\cos h-1)/h=0$, the limit is equal to
$$\lim_{h\to 0}\frac{\sin x_0(\cos h-1)+\cos x_0\sin h}{h}=\sin x_0\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x_0\lim_{h\to 0}\frac{\sin h}{h}=\cos x_0\ .$$

Further Comments on Limits of Functions

First, we have studied limits of functions. For polynomials and rational functions, their limits are well understood. Indeed, let $r(x) = p(x)/q(x)$ be a rational function. We knew (1) it is well defined on the set $E = \{x \in \mathbb{R} : \ q(x) \neq 0\}$, (since a polynomial has at most finitely many roots, E is the union of finitely many open intervals.) (2) $\lim_{x\to x_0} r(x) = r(x_0)$ whenever x_0 satisfies $q(x_0) \neq 0$.

In order to have more examples to work on, we need to introduce more functions. In this chapter the following functions are studied:

- The **square root** $f_1(x) = \sqrt{x}$. It is defined on $[0, \infty)$ and $\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$ for all $x_0 \geq 0$. See Ex 8 for a more general result.
- The **absolute value function** $f_2(x) = |f(x)|$. It is defined on $(-\infty, \infty)$ and $\lim_{x\to x_0} |x| = |x_0|$ for all $x_0 \in (-\infty, \infty)$. See Ex 8 for a more general result.
- The **sine function** $f_3(x) = \sin x, x \in \mathbb{R}$. We do not need a rigorous definition here. Simply assuming that it is an odd function satisfying $x - x^3/6 \leq \sin x \leq x$ for $x \geq 0$, we deduce $\lim_{x\to 0} \sin x/x = 1$.
- The **cosine function** $f_4(x) = \cos x, x \in \mathbb{R}$. Again we do not need a rigorous definition. Simply assuming that it is an even function satisfying $1 - x^2/2 \leq \cos x \leq 1$ for all $x \geq 0$, we deduce $\lim_{x\to 0}(\cos x - 1)/x = 0$ and $\lim_{x\to 0}(\cos x - 1)/x^2 = 1/2$.
- The **exponential function** $f_5(x) = e^x, x \in \mathbb{R}$. When $x \geq 0$, we proved in Ex 5 that $e^x = \lim_{n\to\infty}(1 + x/n)^n = \sum_{n=0}^{\infty} x^n/n!$ and define $e^x = 1/e^{-x}$ for $x < 0$. Assuming that $e^x \geq x$ for $x \geq 0$, we can prove $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$. Also $\lim_{x\to 0^+} e^{1/x} = \infty$ and $\lim_{x\to 0^-} e^{1/x} = 0$.

In the next chapter, we will show that the sine, cosine and exponential functions all satisfy $\lim_{x\to x_0} f(x) = f(x_0)$ for all $x_0 \in \mathbb{R}$, that is, they are continuous everywhere.

Second, divergence at infinity and limits at infinity. Let f be function defined on $(a, b]$. It is said to **diverge to ∞ (resp.** $-\infty$) at a if for each $M > 0$, there is some $\delta > 0$ such that $f(x) > M$ (resp. $f(x) < -M$) for all $x \in (a, a + \delta)$. The notation is $\lim_{x\to a^+} f(x) = \infty$ (resp. $\lim_{x\to a^+} f(x) = -\infty$). Similarly, one can define $\lim_{x\to b^-} f(x) = \pm\infty$. For f defined on (a, ∞) (resp. $(-\infty, b)$) we can define $\lim_{x\to\infty} f(x) = L$ if for each $\varepsilon > 0$ there is some $K > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > K$. Similarly, we can define $\lim_{x\to-\infty} f(x) = L$, $\lim_{x\to\infty} f(x) = \pm\infty$, $\lim_{x\to-\infty} f(x) = \pm\infty$, etc.